

Padé interpolation for elliptic Painlevé equation

Masatoshi Noumi^a, Satoshi Tsujimoto^b and Yasuhiko Yamada^a

a) Department of Mathematics, Faculty of Science,
Kobe University, Hyogo 657-8501, Japan

b) Department of Applied Mathematics and Physics Graduate School of Informatics,
Kyoto University, Kyoto 606-8501, Japan

Dedicated to Professor Michio Jimbo on his 60th birthday

Abstract. An interpolation problem related to the elliptic Painlevé equation is formulated and solved. A simple form of the elliptic Painlevé equation and the Lax pair are obtained. Explicit determinant formulae of special solutions are also given.

1 Introduction

There exists a close connection between the Painlevé equations and the Padé approximations (e.g. [6] [18]). An interesting feature of the Padé approach to Painlevé equation is that we can obtain Painlevé equations, its Lax formalism and special solutions simultaneously once we set up a suitable Padé problem. This method is applicable also for discrete cases and it gave a hint for a Lax pair [19] for the elliptic difference Painlevé equation [14].

In this paper, we analyze the elliptic Painlevé equation, its Lax pair and special solutions, by using the Padé approach. In particular, we study the discrete deformation along one special direction¹. As a result, we obtain remarkably simple form of the elliptic Painlevé equation (39), (40) and its Lax pair (46), (14) or (15), together with their explicit special solutions given by equations (36), (57) and (70).

This paper is organized as follows. In section 2, we set up the interpolation problem. In section 3, we derive two fundamental contiguity relations satisfied by the interpolating functions. In section 4, we show that the variables f, g appearing in the contiguity relations satisfy the elliptic Painlevé equation. Interpretation of the contiguity relations as the Lax pair for elliptic Painlevé equation is given in section 5. In section 6, explicit determinant formulae for the interpolation problem are given. Derivation of the Painlevé equation (39), (40) based on affine Weyl group action is given in Appendix A.

¹ Though all the directions are equivalent due to the Bäcklund transformations, there exists one special direction in the formulation on $\mathbb{P}^1 \times \mathbb{P}^1$ for which the equation take a simple form like QRT system [11]. Jimbo-Sakai's q -Painlevé six equation [3] is a typical example of such beautiful equations. For various q -difference cases, the Lax formalisms for such direction were studied in [20].

2 The interpolation problem

In this section, we will set up an interpolation problem which we study in this paper.

Notations. Let p, q be two base variables satisfying constraints $|p|, |q| < 1$. We denote by $\vartheta_p(x)$ the Jacobi theta function with base p :

$$\vartheta_p(x) = \prod_{i=0}^{\infty} (1 - xp^i)(1 - x^{-1}p^{i+1}), \quad \vartheta_p(px) = \vartheta_p(x^{-1}) = -x^{-1}\vartheta_p(x). \quad (1)$$

The elliptic Gamma function [12] and Pochhammer symbol are defined as

$$\Gamma(x; p, q) = \prod_{i,j=0}^{\infty} \frac{(1 - x^{-1}p^{i+1}q^{j+1})}{(1 - xp^iq^j)}, \quad \vartheta_p(x)_s = \frac{\Gamma(q^s x; p, q)}{\Gamma(x; p, q)} = \prod_{i=0}^{s-1} \vartheta_p(q^i x), \quad (2)$$

where the last equality holds for $s \in \mathbb{Z}_{\geq 0}$. We shall use the standard convention

$$\begin{aligned} \Gamma(x_1, \dots, x_\ell; p, q) &= \Gamma(x_1; p, q) \cdots \Gamma(x_\ell; p, q), \\ \vartheta_p(x_1, \dots, x_\ell)_s &= \vartheta_p(x_1)_s \cdots \vartheta_p(x_\ell)_s. \end{aligned} \quad (3)$$

Padé problem. Let $m, n \in \mathbb{Z}_{\geq 0}$, and let a_1, \dots, a_6, k be complex parameters with a constraint:

$$\prod_{i=1}^6 a_i = k^3. \quad (4)$$

In this paper we consider the following interpolation problem:

$$Y_s = \frac{V(q^{-s})}{U(q^{-s})}, \quad (s = 0, 1, \dots, N = m + n), \quad (5)$$

specified by the following data:

- The interpolated sequence Y_s is given by

$$Y_s = Y(q^{-s}) = \prod_{i=1}^6 \frac{\vartheta_p(a_i)_s}{\vartheta_p(\frac{k}{a_i})_s}, \quad Y(x) = \prod_{i=1}^6 \frac{\Gamma(\frac{a_i}{x}, \frac{k}{a_i}; p, q)}{\Gamma(\frac{k}{a_i x}, a_i; p, q)}. \quad (6)$$

- The interpolating functions $U(x), V(x)$ are defined as

$$U(x) = \sum_{i=0}^n u_i \phi_i(x), \quad V(x) = \sum_{i=0}^m v_i \chi_i(x), \quad (7)$$

with basis

$$\begin{aligned} \phi_i(x) &= \frac{T_{a_2}^{-i} T_{a_4}^i Y(x)}{Y(x)} = \frac{\vartheta_p(\frac{a_4}{x}, \frac{k}{q^i a_4 x})_i}{\vartheta_p(\frac{a_2}{q^i x}, \frac{k}{a_2 x})_i} \frac{\vartheta_p(\frac{a_2}{q^i}, \frac{k}{a_2})_i}{\vartheta_p(a_4, \frac{k}{q^i a_4})_i}, \\ \chi_i(x) &= \frac{Y(x)}{T_{a_1}^i T_{a_3}^{-i} Y(x)} = \frac{\vartheta_p(\frac{a_3}{q^i x}, \frac{k}{a_3 x})_i}{\vartheta_p(\frac{a_1}{x}, \frac{k}{q^i a_1 x})_i} \frac{\vartheta_p(a_1, \frac{k}{q^i a_1})_i}{\vartheta_p(\frac{a_3}{q^i}, \frac{k}{a_3})_i}, \end{aligned} \quad (8)$$

where $T_a : f(a) \mapsto f(qa)$.

The coefficients u_i, v_i are determined by eq.(5) which is linear homogeneous equations. We normalize them as $u_0 = 1$.

Remark on the choice of the bases $\phi_i(x), \chi_i(x)$. The problem we are considering is a version of PPZ scheme (interpolation with prescribed poles and zeros) [21]. Note that

$$\begin{aligned} U(x) &= \frac{U_{\text{num}}(x)}{U_{\text{den}}(x)}, & U_{\text{den}}(x) &= \vartheta_p\left(\frac{a_2}{q^n x}, \frac{k}{a_2 x}\right)_n, \\ V(x) &= \frac{V_{\text{num}}(x)}{V_{\text{den}}(x)}, & V_{\text{den}}(x) &= \vartheta_p\left(\frac{a_1}{x}, \frac{k}{q^m a_1 x}\right)_m, \end{aligned} \quad (9)$$

where $U_{\text{num}}(x), U_{\text{den}}(x)$ (resp. $V_{\text{num}}(x), V_{\text{den}}(x)$) are theta functions of order $2n$ (resp. $2m$). Furthermore, the functions $x^m U_{\text{num}}(x), x^n V_{\text{num}}(x), x^m U_{\text{den}}(x), x^n V_{\text{den}}(x)$ (and hence $U(x), V(x), \phi_i(x), \chi_i(x)$ also) are “symmetric” : $F(k/qx) = F(x)$. We will fix the denominator U_{den} (resp. V_{den}) as above in order to specify the prescribed zeros (resp. poles). For the numerator U_{num} (resp. V_{num}), contrarily, one may take any basis of theta functions as far as they have the same order, same quasi p -periodicity, and same symmetry under $x \leftrightarrow \frac{k}{qx}$ as U_{den} (resp. V_{den}). In this sense, the choice of the basis ϕ_i, χ_i in eq.(8) is not so essential for general argument, however, we will see that it is convenient for explicit expression of the functions $U(x), V(x)$ in section 6.

Parameters of the elliptic Painlevé equation. The elliptic Painlevé equation is specified by a generic configuration of 8 points on $\mathbb{P}^1 \times \mathbb{P}^1$. We parametrize them as $\left(f_*(\xi_i), g_*(\xi_i)\right)_{i=1, \dots, 8}$, where

$$f_*(x) = \frac{\vartheta_p\left(\frac{c_2}{x}, \frac{\kappa_1}{c_2 x}\right)}{\vartheta_p\left(\frac{c_1}{x}, \frac{\kappa_1}{c_1 x}\right)}, \quad g_*(x) = \frac{\vartheta_p\left(\frac{c_4}{x}, \frac{\kappa_2}{c_4 x}\right)}{\vartheta_p\left(\frac{c_3}{x}, \frac{\kappa_2}{c_3 x}\right)}, \quad (10)$$

and c_i are parameters independent of x . The functions $f_*(x), g_*(x)$ satisfy $f_*(x) = f_*\left(\frac{\kappa_1}{x}\right)$, $g_*(x) = g_*\left(\frac{\kappa_2}{x}\right)$, and they give a parametrization of an elliptic curve of degree (2,2).² We define functions $F_f(x)$ and $G_g(x)$ as

$$F_f(x) = \vartheta_p\left(\frac{c_1}{x}, \frac{\kappa_1}{c_1 x}\right)f - \vartheta_p\left(\frac{c_2}{x}, \frac{\kappa_1}{c_2 x}\right), \quad G_g(x) = \vartheta_p\left(\frac{c_3}{x}, \frac{\kappa_2}{c_3 x}\right)g - \vartheta_p\left(\frac{c_4}{x}, \frac{\kappa_2}{c_4 x}\right). \quad (11)$$

Note that $F_f(x) = 0 \Leftrightarrow f = f_*(x)$ and $G_g(x) = 0 \Leftrightarrow g = g_*(x)$.

In this paper, the Painlevé equation appears with the following parameters

$$(\kappa_1, \kappa_2) = \left(k, \frac{k^2}{a_1}\right), \quad (\xi_1, \dots, \xi_8) = \left(\frac{k}{q}, kq^{m+n}, \frac{k}{a_1 q^m}, \frac{a_2}{q^n}, a_3, a_4, a_5, a_6\right). \quad (12)$$

Note that $\kappa_1^2 \kappa_2^2 = q \xi_1 \cdots \xi_8$ due to the constraint (4).

²The choice of parameters c_1, \dots, c_4 (and over all normalization of $f_*(x), g_*(x)$) is related to the fractional linear transformations on $\mathbb{P}^1 \times \mathbb{P}^1$.

3 Contiguity relations

Here, we will derive two fundamental contiguity relations³ satisfied by the functions $V(x)$, $Y(x)U(x)$.

Special direction T of deformation. For any quantity (or function) F depending on variables $k, a_1, \dots, a_6, m, n, \dots$, we denote by $\overline{F} = T(F)$ its parameter shift along a special direction T :

$$T : (k, a_1, \dots, a_6, m, n) \mapsto (kq, \frac{a_1}{q}, a_2, a_3q, \dots, a_6q, m+1, n-1). \quad (13)$$

This special direction is chosen so that $T : (\kappa_1, \kappa_2, \xi_i) \mapsto (\kappa_1q, \kappa_2q^3, \xi_iq)$ and the corresponding elliptic Painlevé equation will take a simple form.

Proposition 1 *The functions $y(x) = V(x), Y(x)U(x)$ satisfy the following contiguity relations:*

$$L_2 : \frac{G_g(\frac{kx}{a_1}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{a_1x}, \frac{k}{qx})} y(x) - \frac{G_g(x) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})}{\vartheta_p(\frac{a_1}{x}, \frac{q}{x})} y(\frac{x}{q}) - \frac{C_0 F_f(x) \vartheta_p(\frac{k}{x^2}, \frac{a_1}{qx}, \frac{kq}{a_1x})}{x} \overline{y}(x) = 0, \quad (14)$$

$$L_3 : G_g(\frac{kqx}{a_1}) \vartheta_p(\frac{k}{qx}, \frac{kq}{a_1x}) \overline{y}(x) - G_g(x) \vartheta_p(\frac{1}{x}, \frac{a_1}{q^2x}) \overline{y}(qx) - \frac{C_1 \overline{F_f(qx)} \vartheta_p(\frac{k}{qx^2})}{x \vartheta_p(\frac{k}{a_1x}, \frac{a_1}{qx})} y(x) = 0, \quad (15)$$

where C_0, C_1, f, g are some constants w.r.t. x .

Proof. We put $\mathbf{y}(x) = \begin{bmatrix} V(x) \\ Y(x)U(x) \end{bmatrix}$ and define the Casorati determinants D_i as

$$\begin{aligned} D_1(x) &:= \det[\mathbf{y}(x), \mathbf{y}(\frac{x}{q})], \\ D_2(x) &:= \det[\mathbf{y}(qx), \mathbf{y}(x)], \\ D_3(x) &:= \det[\overline{\mathbf{y}}(x), \mathbf{y}(x)], \\ D_4(x) &:= \det[\overline{\mathbf{y}}(x), \mathbf{y}(\frac{x}{q})]. \end{aligned} \quad (16)$$

Then the desired contiguity relations are obtained from the identity

$$\begin{aligned} D_1(x) \overline{y}(x) - D_4(x) y(x) + D_3(x) y(\frac{x}{q}) &= 0, \\ D_4(qx) \overline{y}(x) - D_3(x) \overline{y}(qx) - \overline{D_2(x)} y(x) &= 0, \end{aligned} \quad (17)$$

by using the formulae for D_i given in the next Lemma 1. \square

³ Since the contiguity relations (14),(15) are similar to the linear relations of the R_{II} chain [16], it may be possible to derive them as a reduction of three discrete-time non-autonomous Toda chain by using the method in [17].

Lemma 1 *The determinants (16) take the following form:*

$$\begin{aligned}
D_1(x) &= \mathcal{N}(x)Y(x)c \frac{\vartheta_p(\frac{k}{x^2}, \frac{q}{x}, \frac{a_1}{x})F_f(x)}{x\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})}, \\
D_2(x) &= \mathcal{N}(x)Y(x)c \frac{\vartheta_p(\frac{k}{q^2x^2}, \frac{k}{q^2x}, \frac{k}{qxa_1})F_f(qx)}{qx\vartheta_p(\frac{1}{x}, \frac{a_1}{qx}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{qx})}, \\
D_3(x) &= \mathcal{N}(x)Y(x)c' \frac{G_g(x)}{\vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})}, \\
D_4(x) &= \mathcal{N}(x)Y(x)c' \frac{\vartheta_p(\frac{q}{x}, \frac{a_1}{x})G_g(\frac{kx}{a_1})}{\vartheta_p(\frac{k}{qx}, \frac{k}{qx}, \frac{k}{xa_1}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx})} \prod_{i=1}^8 \frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{x\xi_i})},
\end{aligned} \tag{18}$$

where

$$\mathcal{N}(x) = \frac{\vartheta_p(\frac{1}{q^{m+n}x}, \frac{k}{qx})_{m+n+1}}{U_{\text{den}}(x)V_{\text{den}}(x)}. \tag{19}$$

Proof. The functions $U(x)$, $V(x)$ and, due to the constraint (4), the function $Y(x)$ are elliptic (p -periodic) functions in x . Hence the ratios $\frac{D_i(x)}{Y(x)}$ are also elliptic. They are of order $2m + 2n + (\text{small corrections})$ and have sequences of zeros and poles represented as $\vartheta_p(\frac{1}{q^{m+n}x}, \frac{k}{qx})_{m+n+1}$ and $U_{\text{den}}V_{\text{den}}$ modulo corrections at the boundaries of the sequence. Then we can compute the ratios $\frac{D_i(x)}{Y(x)}$, and each of them are determined up to 2 unknown constants. In the computation, the following relations are useful (they are derived by a straightforward computation)

$$G(x) := \frac{Y(qx)}{Y(x)} = \prod_{i=1}^6 \frac{\vartheta_p(\frac{k}{a_i qx})}{\vartheta_p(\frac{a_i}{qx})}, \tag{20}$$

$$K(x) := \frac{\overline{Y}(x)}{Y(x)} = \frac{\vartheta_p(\frac{k}{a_1}, \frac{k}{a_2}, \frac{a_1}{q}, \frac{kq}{a_1})}{\vartheta_p(\frac{k}{a_1x}, \frac{k}{a_2x}, \frac{a_1}{qx}, \frac{kq}{a_1x})} \prod_{i=3}^6 \frac{\vartheta_p(\frac{a_i}{x})}{\vartheta_p(a_i)}, \tag{21}$$

$$\mathcal{N}(\frac{k}{qx}) = \frac{qx^2}{k} \mathcal{N}(x), \tag{22}$$

and

$$\frac{\mathcal{N}(qx)}{\mathcal{N}(x)} = \frac{\vartheta_p(\frac{q}{x}, \frac{q^N k}{x}, \frac{a_1}{x}, \frac{k}{q^m a_1 x}, \frac{k}{a_2 x}, \frac{a_2}{q^N x})}{\vartheta_p(\frac{1}{q^{N+1}x}, \frac{k}{qx}, \frac{q^m a_1}{x}, \frac{k}{a_1 x}, \frac{q^N k}{a_2 x}, \frac{a_2}{x})}. \tag{23}$$

• *Computation of $D_1(x)$, $D_2(x)$:* First, we count the degree of the elliptic function

$$\frac{D_1(x)}{Y(x)} = \frac{1}{G(\frac{x}{q})} V(x) U(\frac{x}{q}) - V(\frac{x}{q}) U(x). \tag{24}$$

Substituting

$$\begin{aligned}
U(\frac{x}{q}) &= \frac{U_{\text{num}}(\frac{x}{q})}{U_{\text{den}}(\frac{x}{q})} = \frac{\vartheta_p(\frac{k}{a_2 x}, \frac{a_2}{q^N x}) U_{\text{num}}(\frac{x}{q})}{\vartheta_p(\frac{q^N k}{a_2 x}, \frac{a_2}{x}) U_{\text{den}}(x)}, \\
V(\frac{x}{q}) &= \frac{V_{\text{num}}(\frac{x}{q})}{V_{\text{den}}(\frac{x}{q})} = \frac{\vartheta_p(\frac{k}{q^m a_2 x}, \frac{a_1}{x}) V_{\text{num}}(\frac{x}{q})}{\vartheta_p(\frac{k}{a_1 x}, \frac{q^m a_1}{x}) V_{\text{den}}(x)},
\end{aligned} \tag{25}$$

we have

$$\begin{aligned} \frac{D_1(x)}{Y(x)} &= \frac{1}{U_{\text{den}}(x)V_{\text{den}}(x)} \frac{\vartheta_p(\frac{a_1}{x})}{\vartheta_p(\frac{k}{a_1x})} \\ &\times \left\{ \frac{\vartheta_p(\frac{a_2}{q^n x})}{\vartheta_p(\frac{q^n k}{a_2 x})} \prod_{i=3}^6 \frac{\vartheta_p(\frac{a_i}{x})}{\vartheta_p(\frac{k}{a_i x})} V_{\text{num}}(x) U_{\text{num}}(\frac{x}{q}) - \frac{\vartheta_p(\frac{k}{q^m a_1 x})}{\vartheta_p(\frac{q^m a_1}{x})} U_{\text{num}}(x) V_{\text{num}}(\frac{x}{q}) \right\}. \end{aligned} \quad (26)$$

The function $D_1(x)/Y(x)$ is p -periodic function of order $2m + 2n + 6$ with denominator

$$U_{\text{den}}(x) \left\{ V_{\text{den}}(x) \frac{\vartheta_p(\frac{k}{a_1 x})}{\vartheta_p(\frac{a_1}{x})} \right\} \vartheta_p(\frac{q^m a_1}{x}) \vartheta_p(\frac{q^n k}{a_2 x}) \prod_{i=3}^6 \vartheta_p(\frac{k}{a_i x}). \quad (27)$$

Next, we study the zeros. When x and $\frac{x}{q}$ are both in the Padé interpolation grid (i.e. for $x = 1, q^{-1}, \dots, q^{-N+1}$), it follows obviously that $D_1(x) = 0$. Noting the symmetry properties

$$U(\frac{k}{qx}) = U(x), \quad V(\frac{k}{qx}) = V(x), \quad G(\frac{k}{qx}) = \frac{1}{G(\frac{x}{q})}, \quad (28)$$

we have

$$\frac{D_1(\frac{k}{x})}{Y(\frac{k}{x})} = G(\frac{x}{q}) U(x) V(\frac{x}{q}) - U(\frac{x}{q}) V(x) = -G(\frac{x}{q}) \frac{D_1(x)}{Y(x)}. \quad (29)$$

Then it follows that $D_1(x) = 0$ at $x = k, kq, \dots, kq^{N-1}$ and furthermore, due to the relation $\mathbf{y}(x) = \mathbf{y}(\frac{x}{q})$ for $x^2 = k$, we have $D_1(x) = 0$ at $x^2 = k$ (i.e. $x = \pm\sqrt{k}, \pm\sqrt{kq}$). As a result, the function $X(x)$ defined by

$$D_1(x) = \mathcal{N}(x) Y(x) \frac{\vartheta_p(\frac{k}{x^2}, \frac{q}{x}, \frac{a_1}{x})}{x \vartheta_p(\frac{k}{qx}, \frac{k}{xa_1}) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})} X(x) \quad (30)$$

is a theta function of degree 2 such that $X(\frac{x}{p}) = X(\frac{k}{x}) = \frac{x^2}{k} X(x)$, hence it can be written as $X(x) = cF_f(x)$ by suitable constants c, f . D_2 is easily obtained since $D_2(x) = D_1(qx)$.

• *Computation of $D_3(x), D_4(x)$* : First we note a relation between $D_3(x)$ and $D_4(x)$. Using $U(\frac{k}{qx}) = U(x)$, $\bar{U}(\frac{k}{x}) = \bar{U}(x)$ and similar relations for $V(x)$ we have

$$\begin{aligned} \frac{D_3(\frac{k}{qx})}{Y(\frac{k}{qx})} &= U(\frac{k}{qx}) \bar{V}(\frac{k}{qx}) - K(\frac{k}{qx}) \bar{U}(\frac{k}{qx}) V(\frac{k}{qx}) \\ &= U(x) \bar{V}(qx) - K(\frac{k}{qx}) \bar{U}(qx) V(x) \\ &= \frac{G(x)}{Y(qx)} \left\{ Y(x) U(x) \bar{V}(qx) - \frac{K(\frac{k}{qx})}{G(x)} Y(qx) \bar{U}(qx) V(x) \right\} \\ &= G(x) \frac{D_4(qx)}{Y(qx)}, \end{aligned} \quad (31)$$

where we have used the relation $\frac{K(\frac{k}{qx})}{G(x)} = K(qx)$ at the last step.

Let us compute $D_3(x)$. Substituting the relation

$$\begin{aligned}\overline{U}(x) &= \frac{\overline{U_{\text{num}}}(x)}{\overline{U_{\text{den}}}(x)} = \vartheta_p\left(\frac{k}{a_2x}, \frac{a_2}{q^n x}\right) \frac{\overline{U_{\text{num}}}(x)}{\overline{U_{\text{den}}}(x)}, \\ \overline{V}(x) &= \frac{\overline{V_{\text{num}}}(x)}{\overline{V_{\text{den}}}(x)} = \frac{\vartheta_p\left(\frac{k}{q^m a_1 x}\right)}{\vartheta_p\left(\frac{a_1}{qx}, \frac{k}{a_1 x}, \frac{qk}{a_1 x}\right)} \frac{\overline{V_{\text{num}}}(x)}{\overline{V_{\text{den}}}(x)},\end{aligned}\tag{32}$$

into

$$\frac{D_3(x)}{Y(x)} = U(x)\overline{V}(x) - K(x)\overline{U}(x)V(x),\tag{33}$$

we have

$$\begin{aligned}\frac{D_3(x)}{Y(x)} &= \frac{1}{U_{\text{den}}(x)V_{\text{den}}(x)} \frac{1}{\vartheta_p\left(\frac{k}{a_1 x}, \frac{qk}{a_1 x}, \frac{a_1}{qx}\right)} \\ &\quad \times \left\{ \vartheta_p\left(\frac{a_2}{q^n x}, \frac{a_3}{x}, \dots, \frac{a_6}{x}\right) V_{\text{num}}(x)\overline{U_{\text{num}}}(x) - \vartheta_p\left(\frac{k}{q^m a_1 x}\right) \overline{V_{\text{num}}}(x)U_{\text{num}}(x) \right\},\end{aligned}\tag{34}$$

Hence, $\frac{D_3(x)}{Y(x)}$ is of degree $2m + 2n + 3$.

$D_3(x)$ has zeros at $x = 1, q^{-1}, \dots, q^{-N}$ and $x = k, qk, \dots, q^{N-1}k$, where the latter zeros follow from those of $D_4(x)$ through eq.(31). Hence, we obtain

$$D_3(x) = \mathcal{N}(x)Y(x) \frac{1}{\vartheta_p\left(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx}\right)} Z(x),\tag{35}$$

where $Z(x)$ is a theta function of degree 2 such as $Z(\frac{x}{p}) = Z(\frac{k^2}{a_1 x}) = \frac{a_1 x^2}{k^2} Z(x)$, namely $Z(x) = c' G_g(x)$ for some c' and g as desired. $D_4(x)$ is derived by the relation (31). \square

Corollary 1 *For any pair $i, j \in \{3, 4, 5, 6\}$ we have*

$$\frac{\alpha(a_i)}{\alpha(a_j)} \frac{F_f(a_i)}{F_f(a_j)} = \frac{U(a_i)V(a_i/q)}{U(a_j)V(a_j/q)}, \quad \frac{\beta(a_i)}{\beta(a_j)} \frac{G_g(a_i)}{G_g(a_j)} = \frac{U(a_i)\overline{V}(a_i)}{U(a_j)\overline{V}(a_j)},\tag{36}$$

where

$$\alpha(x) = \mathcal{N}(x) \frac{\vartheta_p\left(\frac{k}{x^2}, \frac{q}{x}, \frac{a_1}{x}\right)}{x \vartheta_p\left(\frac{k}{qx}, \frac{k}{xa_1}\right) \prod_{i=1}^8 \vartheta_p\left(\frac{k}{x\xi_i}\right)}, \quad \beta(x) = \mathcal{N}(x) \frac{1}{\vartheta_p\left(\frac{k}{qx}, \frac{k}{xa_1}, \frac{kq}{xa_1}, \frac{a_1}{qx}\right)}.\tag{37}$$

Proof. By the definition of D_1, D_3 , we have for $x = a_i$ ($i = 3, 4, 5, 6$)

$$\begin{aligned}\frac{D_1(x)}{Y(x)} &= \frac{1}{G(x/q)} V(x)U\left(\frac{x}{q}\right) - U(x)V\left(\frac{x}{q}\right) = -U(x)V\left(\frac{x}{q}\right), \\ \frac{D_3(x)}{Y(x)} &= \overline{V}(x)U(x) - K(x)\overline{U}(x)V(x) = U(x)\overline{V}(x).\end{aligned}\tag{38}$$

Then, from the first and the third equation of (18), one has eq.(36). \square

The formulae (36) are convenient in order to obtain f, g from $U(x), V(x)$.

4 Elliptic Painlevé equation

In this section, we study the eqs.(14),(15) for generic variables f, g apart from the Padé problem, and prove that the variables f, g satisfy the elliptic Painlevé equation.

Theorem 1 *If the eqs.(14),(15) are compatible, then the variables f, g and \bar{f}, \bar{g} should be related by*

$$\frac{F_f(x)\overline{F_f(qx)}}{F_f(\frac{xa_1}{k})\overline{F_f(\frac{q^2xa_1}{k})}} = \prod_{i=1}^8 \frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k^2}{x\xi_ia_1})}, \quad \text{for } g = g_*(x), \quad (39)$$

and

$$\frac{G_g(x)\overline{G_g(qx)}}{G_g(\frac{kqx}{a_1})\overline{G_g(\frac{kqx}{a_1})}} = \prod_{i=1}^8 \frac{\vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx\xi_i})}, \quad \text{for } \bar{f} = \bar{f}_*(qx). \quad (40)$$

Proof. From equations $\overline{L_2}|_{x \rightarrow qx}$ (14) and L_3 (15) we have

$$\begin{aligned} \frac{\overline{G_g(\frac{kqx}{a_1})} \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{kq}{a_1x}, \frac{k}{qx})} \bar{y}(qx) &= \frac{\overline{G_g(qx)} \prod_{i=1}^8 \vartheta_p(\frac{k}{qx\xi_i})}{\vartheta_p(\frac{a_1}{q^2x}, \frac{1}{x})} \bar{y}(x), \\ G_g(\frac{kqx}{a_1}) \vartheta_p(\frac{k}{qx}, \frac{kq}{a_1x}) \bar{y}(x) &= G_g(x) \vartheta_p(\frac{1}{x}, \frac{a_1}{q^2x}) \bar{y}(qx), \end{aligned} \quad (41)$$

for $\bar{f} = \bar{f}_*(qx)$, hence we have eq.(40).

For $g = g_*(x)$, we have from eqs.(14), (15) that

$$\begin{aligned} \frac{G_g(\frac{kx}{a_1}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{a_1x}, \frac{k}{qx})} y(x) &= \frac{C_0 F_f(x) \vartheta_p(\frac{k}{x^2}, \frac{a_1}{qx}, \frac{kq}{a_1x})}{x} \bar{y}(x), \\ G_g(\frac{kqx}{a_1}) \vartheta_p(\frac{k}{qx}, \frac{kq}{a_1x}) \bar{y}(x) &= \frac{C_1 \overline{F_f(qx)} \vartheta_p(\frac{k}{qx^2})}{x \vartheta_p(\frac{k}{a_1x}, \frac{a_1}{qx})} y(x), \end{aligned} \quad (42)$$

hence

$$G_g(\frac{kx}{a_1}) G_g(\frac{kqx}{a_1}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x}) = \frac{w}{x^2} F_f(x) \overline{F_f(qx)} \vartheta_p(\frac{k}{x^2}, \frac{k}{qx^2}), \quad (43)$$

where $w = C_0 C_1$. The eq.(43) holds also by replacing $x \rightarrow \frac{k^2}{a_1x}$ since $g_*(x) = g_*(\frac{k^2}{a_1x})$. Taking a ratio eq.(43) with eq.(43)| $_{x \rightarrow \frac{k^2}{a_1x}}$ we have eq.(39). \square

The next Lemma 2 shows that the relations (39),(40) are equivalent to the time evolution equation for the elliptic Painlevé.⁴

Lemma 2 *The solution \bar{f} of eq.(39) is a rational function of (f, g) of degree $(1, 4)$, which is characterized by the following conditions: (i) its numerator and denominator have 8 zeros at $f = f_*(\xi), g = g_*(\xi)$, (ii) if $f = f_*(u), g = g_*(u)$ ($u \neq \xi$) then $\bar{f} = \bar{f}_*(\frac{a_1u}{k})$. Similarly, by eq.(40), \bar{g} is uniquely given as a rational function of (\bar{f}, g) of degree $(4, 1)$, satisfying the conditions (i') it has 8 points of indeterminacy at $\bar{f} = \bar{f}_*(q\xi), g = g_*(\xi)$, (ii') if $\bar{f} = \bar{f}_*(qu), g = g_*(u)$ ($u \neq \xi$) then $\bar{g} = \bar{g}_*(\frac{q^3ku}{a_1}) = g_*(\frac{qku}{a_1})$.*

⁴ Since the elliptic Painlevé equation [14] is rather complicated, its concise expressions have been pursued by several authors (e.g. [8],[9],[10]). The system (39),(40) is supposed to be the simplest one.

Proof. Written in the form

$$F_f(x) \overline{F_f(qx)} \prod_{i=1}^8 \vartheta_p\left(\frac{k^2}{x \xi_i a_1}\right) = F_f\left(\frac{x a_1}{k}\right) \overline{F_f\left(\frac{q^2 x a_1}{k}\right)} \prod_{i=1}^8 \vartheta_p\left(\frac{\xi_i}{x}\right), \quad (44)$$

the eq.(39) is quasi p -periodic in x of degree (apparently) 12 with symmetry under $x \leftrightarrow \frac{k^2}{a_1 x}$. Since it is divisible by a factor $\vartheta_p(\frac{k^2}{a_1 x^2})$, it is effectively of degree 8. Then the solution \overline{f} of this equation takes the form

$$\overline{f} = \frac{A(x)f + B(x)}{C(x)f + D(x)}, \quad (45)$$

where the coefficients $A(x), \dots, D(x)$ are $x \leftrightarrow \frac{k^2}{a_1 x}$ -symmetric p -periodic functions of degree 8, namely polynomials of $g = g_*(x)$ of degree 4. Hence \overline{f} is a rational function of (f, g) of degree (1, 4). The conditions (i), (ii) are obvious by the form of eq.(39). The structure of the solution $\overline{g} = \overline{g}(\overline{f}, g)$ of the eq.(40) is similar. \square

Remark on the geometric characterization of the solutions f, g . As a consequence of the above results, the variables f, g obtained from the Padé problem give special solutions of the elliptic Painlevé equation. Since they are (Bäcklund transformations of) the terminating hypergeometric solution [4] [5], they have the following geometric characterization.

Let C_1 be a curve of degree $(2n, 2n+1)$ passing through the 8 points $\left(f_*(\xi_i), g_*(\xi_i)\right)_{i=1}^8$ in eqs.(10),(12) with multiplicity $n(1^8) + (0, 1, 1, 0, 0, 0, 0, 0)$. Similarly, Let C_2 be a curve of degree $(2m+2, 2m+1)$ passing through the 8 points with multiplicity $m(1^8) + (0, 1, 0, 1, 1, 1, 1, 1)$. C_1 and C_2 are unique rational curves. Except for the assigned 8 points, there exist unique unassigned intersection point $(f, g) \in C_1 \cap C_2$ which is the solution.

5 Lax formalism

In this section, we prove that the elliptic Painlevé equation (39),(40) are sufficient for the compatibility of eqs.(14),(15).

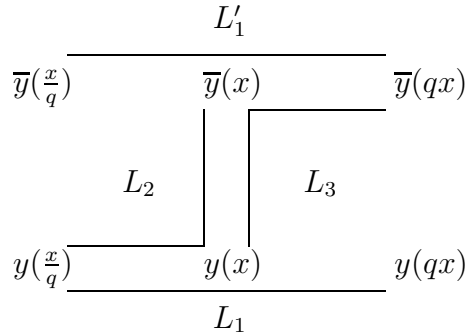


Figure 1: Lax equations

Solving $\bar{y}(x)$ and $\bar{y}(qx)$ from eqs. L_2 , $L_2|_{x \rightarrow qx}$ and plugging them into L_3 , one has the following difference equation (Fig.1):

$$L_1 : \frac{\vartheta_p(\frac{k}{a_1x}, \frac{k}{qx}) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})}{F_f(x)\vartheta_p(\frac{k}{x^2}, \frac{a_1}{x}, \frac{q}{x})} y(\frac{x}{q}) + \frac{q\vartheta_p(\frac{1}{x}, \frac{a_1}{qx}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{qx})}{F_f(qx)\vartheta_p(\frac{k}{q^2x^2}, \frac{k}{q^2x}, \frac{k}{a_1qx})} y(qx) \\ + \left\{ \frac{wF_f(qx)\vartheta_p(\frac{k}{qx^2})}{x^2G_g(x)G_g(\frac{kqx}{a_1})} - \frac{qG_g(qx) \prod_{i=1}^8 \vartheta_p(\frac{k}{qx\xi_i})}{F_f(qx)G_g(\frac{kqx}{a_1})\vartheta_p(\frac{k}{q^2x^2})} - \frac{G_g(\frac{kx}{a_1}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{F_f(x)G_g(x)\vartheta_p(\frac{k}{x^2})} \right\} y(x) = 0. \quad (46)$$

The pairs of equations $\{L_1, L_2\}$, $\{L_1, L_3\}$ and $\{L_2, L_3\}$ are equivalent with each other.

The above expression L_1 (46) contains variables f, g, \bar{f}, w . We will rewrite and characterize it in terms of f, g only. This characterization is a key of the proof of the compatibility. To do this, we first note the following

Lemma 3 *The factor w satisfying the relation (43) is explicitly given by (f, g) as*

$$w = C \frac{\bar{f}_{\text{den}}(f, g)}{\varphi(f, g)}, \quad (47)$$

where $\bar{f}_{\text{den}}(f, g)$ is a polynomial of degree $(1, 4)$ defined as the denominator of the rational function $\bar{f} = \bar{f}(f, g)$, and $\varphi(f, g)$ is the defining polynomial of the degree $(2, 2)$ curve parametrized by $f_*(x), g_*(x)$, and C is a constant independent of f, g, x .

Proof. The relation (43) follows from eq.(47) by using

$$\varphi|_{g=g_*(x)} = C' \frac{F_f(x)F_f(\frac{a_1x}{k})}{g_{*\text{den}}(x)^2}, \quad (48)$$

$$\left(\bar{f}_{\text{den}}\bar{f}_{*\text{num}}(qx) - \bar{f}_{\text{num}}\bar{f}_{*\text{den}}(qx) \right) \Big|_{g=g_*(x)} = C'' \frac{F_f(\frac{a_1x}{k}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{g_{*\text{den}}(x)^4}, \quad (49)$$

where C', C'' are constants, $g_{*\text{den}}(x) = \vartheta_p(\frac{c_3}{x}, \frac{k^2}{a_1c_3x})$ is the denominator of $g_*(x)$, and similarly $\bar{f}_{*\text{den}}(x) = \vartheta_p(\frac{c_1}{x}, \frac{kq}{c_1x})$, $\bar{f}_{*\text{num}}(x) = \vartheta_p(\frac{c_2}{x}, \frac{kq}{c_2x})$. \square

Lemma 4 *In terms of variables f, g , the eq.(46) is represented as a polynomial equation $L_1(f, g) = 0$ of degree $(3, 2)$ characterized⁵ by the following vanishing conditions at: (1) 10 points $(f_*(u), g_*(u))$ where $u = \xi, qx$ and $\frac{k}{x}$, (2) 2 more points (f, g) such as*

$$f = f_*(x), \quad \frac{y(x)}{y(\frac{x}{q})} \frac{G_g(\frac{kx}{a_1})}{G_g(x)} = \frac{\vartheta_p(\frac{k}{a_1x}, \frac{k}{qx})}{\vartheta_p(\frac{a_1}{x}, \frac{q}{x})} \prod_{i=1}^8 \frac{\vartheta_p(\frac{k}{\xi_i x})}{\vartheta_p(\frac{\xi_i}{x})}, \quad (50)$$

and

$$f = f_*(qx), \quad \frac{y(qx)}{y(x)} \frac{G_g(\frac{kqx}{a_1})}{G_g(qx)} = \frac{\vartheta_p(\frac{k}{a_1qx}, \frac{k}{q^2x})}{\vartheta_p(\frac{a_1}{qx}, \frac{1}{x})} \prod_{i=1}^8 \frac{\vartheta_p(\frac{k}{\xi_i qx})}{\vartheta_p(\frac{\xi_i}{qx})}. \quad (51)$$

⁵This geometric characterization of the difference equation L_1 is essentially the same as that in [19].

Proof. Due to the eq.(43), the residue of L_1 at the apparent pole $g = g_*(x)$ vanishes. Replacing x with $\frac{k}{qx}$ in eq.(43) and using the relations $F_f(\frac{k}{x}) = \frac{x^2}{k}F_f(x)$ and $G_g(\frac{k^2}{a_1x}) = \frac{a_1x^2}{k^2}G_g(x)$, we have

$$qx^2G_g(x)G_g(qx)\prod_{i=1}^8\vartheta_p\left(\frac{k}{q\xi_i x}\right) = wF_f(qx)\overline{F_f(qx)}\vartheta_p\left(\frac{k}{qx^2}, \frac{k}{q^2x^2}\right), \quad (52)$$

hence, the residue of L_1 at $g = g_*(\frac{kqx}{a_1}) = g_*(\frac{k}{qx})$ also vanishes. From these vanishing of residues and the eq.(47), the L.H.S of eq.(46) turns out to be a polynomial in (f, g) of degree $(3, 2)$, after multiplying by $F_f(x)F_f(qx)\varphi$. Check of the vanishing conditions (1),(2) are easy. \square

In a similar way, solving $y(\frac{x}{q}), y(x)$ form $L_3, L_3|_{x \rightarrow x/q}$ and substituting them into L_2 , one has

$$\begin{aligned} L'_1 : & \frac{\vartheta_p(\frac{1}{x}, \frac{a_1}{q^2x}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx^2}, \frac{k}{qx}, \frac{kq}{xa_1}) \overline{F_f(qx)}} \overline{y}(qx) + \frac{\vartheta_p(\frac{k}{x}, \frac{kq^2}{xa_1}) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})}{q\vartheta_p(\frac{kq}{x^2}, \frac{q}{x}, \frac{a_1}{qx}) \overline{F_f(x)}} \overline{y}(\frac{x}{q}) \\ & + \left\{ \frac{w\vartheta_p(\frac{k}{x^2})F_f(x)}{x^2G_g(x)G_g(\frac{kx}{a_1})} - \frac{G_g(\frac{x}{q}) \prod_{i=1}^8 \vartheta_p(\frac{k}{x\xi_i})}{q\vartheta_p(\frac{kq}{x^2}) \overline{F_f(x)}G_g(\frac{kx}{a_1})} - \frac{G_g(\frac{kqx}{a_1}) \prod_{i=1}^8 \vartheta_p(\frac{\xi_i}{x})}{\vartheta_p(\frac{k}{qx^2}) \overline{F_f(qx)}G_g(x)} \right\} \overline{y}(x) = 0. \end{aligned} \quad (53)$$

By the similar analysis as L_1 , we have the following

Lemma 5 *In terms of variables \overline{f}, g , the eq.(53) is represented as a polynomial equation $L'_1(\overline{f}, g) = 0$ of degree $(3, 2)$ characterized by the following vanishing conditions at: (1) 10 points $(\overline{f}_*(qu), g_*(u))$ where $u = \xi, \frac{x}{q}$ and $\frac{k}{qx}$. (2) 2 more points (\overline{f}, g) such as*

$$\overline{f} = \overline{f_*(x)}, \quad \frac{\overline{y}(x)}{\overline{y}(\frac{x}{q})} \frac{G_g(\frac{x}{q})}{G_g(\frac{kx}{a_1})} = \frac{\vartheta_p(\frac{kq^2}{a_1x}, \frac{k}{x})}{\vartheta_p(\frac{a_1}{qx}, \frac{q}{x})}, \quad (54)$$

and

$$\overline{f} = \overline{f_*(qx)}, \quad \frac{\overline{y}(qx)}{\overline{y}(x)} \frac{G_g(x)}{G_g(\frac{kqx}{a_1})} = \frac{\vartheta_p(\frac{kq}{a_1x}, \frac{k}{qx})}{\vartheta_p(\frac{a_1}{qx}, \frac{1}{x})}. \quad (55)$$

Proof. In terms of (\overline{f}, g) , the gauge factor w (47) is written as

$$w = C''' \frac{f_{\text{den}}(\overline{f}, g)}{\overline{\varphi}(\overline{f}, g)}, \quad (56)$$

where $f_{\text{den}}(\overline{f}, g)$ is the denominator of the rational function $f = f(\overline{f}, g)$, and $\overline{\varphi}(\overline{f}, g)$ is the defining polynomial of the curve parametrized by $\overline{f}_*(qx), g_*(x)$, and C''' is a constant. Then the proof of the Lemma is the same as the proof of the Lemma 4. \square

Proposition 2 *The eq.(53) expressed in terms of $(\overline{f}, \overline{g})$ is equivalent with the transformation $T(L_1) = \overline{L_1}$ of eq.(46).*

Proof. This fact is a consequence of Lemmas 2, 4 and 5. The geometric proof in the q -difference case [20] is also available here (see Lemmas 4.2 - 4.6 in [20]). \square

6 Determinant formulae

In this section, we present explicit determinant formulae for the solutions $U(x)$, $V(x)$ of the interpolation problem (5).

Theorem 2 *Interpolating rational functions $U(x)$, $V(x)$ have the following determinant expressions:*

$$U(x) = \text{const.} \begin{vmatrix} m_{0,0}^U & \cdots & m_{0,n}^U \\ \vdots & \ddots & \vdots \\ m_{n-1,0}^U & \cdots & m_{n-1,n}^U \\ \phi_0(x) & \cdots & \phi_n(x) \end{vmatrix}, \quad V(x) = \text{const.} \begin{vmatrix} m_{0,0}^V & \cdots & m_{0,m}^V \\ \vdots & \ddots & \vdots \\ m_{m-1,0}^V & \cdots & m_{m-1,m}^V \\ \chi_0(x) & \cdots & \chi_m(x) \end{vmatrix}, \quad (57)$$

where

$$\begin{aligned} m_{ij}^U &= {}_{12}V_{11}(q^{-1}k, q^{-N}, q^{N-i-1}a_1, q^{-j}a_2, q^i a_3, q^j a_4, a_5, a_6; q), \\ m_{ij}^V &= {}_{12}V_{11}(q^{-1}k, q^{-N}, q^{-j} \frac{k}{a_1}, q^{N-i-1} \frac{k}{a_2}, q^j \frac{k}{a_3}, q^i \frac{k}{a_4}, \frac{k}{a_5}, \frac{k}{a_6}; q), \end{aligned} \quad (58)$$

and ${}_{n+5}V_{n+4}$ (${}_{n+3}E_{n+2}$ in convention of [4]) is the very-well poised, balanced elliptic hypergeometric series [1][15]:

$${}_{n+5}V_{n+4}(u_0; u_1, \dots, u_n; z) = \sum_{s=0}^{\infty} \frac{\vartheta_p(u_0 q^{2s})}{\vartheta_p(u_0)} \prod_{j=0}^n \frac{\vartheta_p(u_j)_s}{\vartheta_p(qu_0/u_j)_s} z^s. \quad (59)$$

Proof. In general, the solution of interpolation problem

$$V(x_s) = Y_s U(x_s), \quad s = 0, \dots, N. \quad (60)$$

is written by the following determinants:

$$U(x) = \begin{vmatrix} \chi_0(x_0) & \cdots & \chi_m(x_0) & Y_0 \phi_0(x_0) & \cdots & Y_0 \phi_n(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \chi_0(x_N) & \cdots & \chi_m(x_N) & Y_N \phi_0(x_N) & \cdots & Y_N \phi_n(x_N) \\ 0 & \cdots & 0 & \phi_0(x) & \cdots & \phi_n(x) \end{vmatrix}, \quad (61)$$

and

$$V(x) = \begin{vmatrix} \chi_0(x_0) & \cdots & \chi_m(x_0) & Y_0 \phi_0(x_0) & \cdots & Y_0 \phi_n(x_0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \chi_0(x_N) & \cdots & \chi_m(x_N) & Y_N \phi_0(x_N) & \cdots & Y_N \phi_n(x_N) \\ \chi_0(x) & \cdots & \chi_m(x) & 0 & \cdots & 0 \end{vmatrix}. \quad (62)$$

We apply these formulae for Y_s , $\phi_i(x)$, $\chi_i(x)$ given by (6), (8) and $x_s = q^{-s}$. Note that $\phi_i(x_s)$, $\chi_i(x_s)$ can be written as

$$\begin{aligned} \phi_i(x_s) &= \frac{\vartheta_p(\frac{k}{a_2}, \frac{k}{a_4}, q^{-i}a_2, q^i a_4)_s}{\vartheta_p(a_2, a_4, q^i \frac{k}{a_2}, q^{-i} \frac{k}{a_4})_s}, \\ \chi_i(x_s) &= \frac{\vartheta_p(a_1, a_3, q^{-i} \frac{k}{a_1}, q^i \frac{k}{a_3})_s}{\vartheta_p(\frac{k}{a_1}, \frac{k}{a_3}, q^i a_1, q^{-i} a_3)_s}. \end{aligned} \quad (63)$$

To rewrite the determinant in eq.(61), we use the multiplication by a matrix

$$L = \left[\frac{(L_{ij})_{i,j=0}^N}{1} \right] \quad (64)$$

from the left, where

$$L_{ij} = \frac{\vartheta_p(q^{2j-1}k)}{\vartheta_p(q^{-1}k)} \frac{\vartheta_p(q^{-1}k, q^{-N}, q^{N-i-1}a_1, \frac{k}{a_1}, q^i a_3, \frac{k}{a_3})_j}{\vartheta_p(q, q^N k, q^{-N+i+1} \frac{k}{a_1}, a_1, q^{-i} \frac{k}{a_3}, a_3)_j} q^j. \quad (65)$$

For the last $n+1$ columns, we have

$$\sum_{s=0}^N L_{is} Y_s \phi_j(x_s) = {}_{12}V_{11}(q^{-1}k; q^{-N}, q^{N-i-1}a_1, q^{-j}a_2, q^i a_3, q^j a_4, a_5, a_6; q) = m_{ij}^U. \quad (66)$$

For the first $m+1$ columns, we have

$$\sum_{s=0}^N L_{is} \chi_j(x_s) = {}_{10}V_9(q^{-1}k; q^{-N}, q^{N-i-1}a_1, q^{-j} \frac{k}{a_1}, q^i a_3, q^j \frac{k}{a_3}; q). \quad (67)$$

Using the Frenkel-Turaev summation formula ($u_1 \cdots u_5 = qu_0^2$, $u_5 = q^{-n}$) [2][15]:

$${}_{10}V_9(u_0; u_1, \dots, u_5; q) = \frac{\vartheta_p(qu_0, \frac{qu_0}{u_1 u_2}, \frac{qu_0}{u_1 u_3}, \frac{qu_0}{u_2 u_3})_n}{\vartheta_p(\frac{qu_0}{u_1}, \frac{qu_0}{u_2}, \frac{qu_0}{u_3}, \frac{qu_0}{u_1 u_2 u_3})_n}, \quad (68)$$

the expression (67) can be evaluated as

$$\frac{\vartheta_p(k, q^{-N+i+j+1}, q^{-N+1} \frac{k}{a_1 a_3}, q^{j-i} \frac{a_1}{a_3})_N}{\vartheta_p(q^{-N+j+1} \frac{1}{a_3}, q^{-i} \frac{k}{a_3}, q^j a_1, q^{-N+i+1} \frac{k}{a_1})_N}, \quad (69)$$

and it vanishes for $0 \leq i+j < N$. Hence, we obtain the formula for $U(x)$ in (57) by Laplace expansion. The case for function $V(x)$ is similar. \square

Theorem 2 supplies also formulae for special solutions f, g of the elliptic Painlevé equation through eq.(36). Moreover we have

Lemma 6 For $i, j \in \{3, 4, 5, 6\}$, the ratios in eq.(36) have following simple form

$$\frac{U(a_i)}{U(a_j)} = \frac{c_i T_{a_2}^{-1} T_{a_i}(\tau^U)}{c_j T_{a_2}^{-1} T_{a_j}(\tau^U)}, \quad \frac{V(a_i/q)}{V(a_j/q)} = \frac{c'_i T_{a_1} T_{a_i}^{-1}(\tau^V)}{c'_j T_{a_1} T_{a_j}^{-1}(\tau^V)}, \quad (70)$$

where $\tau^U = \det(m_{i,j}^U)_{i,j=0}^{n-1}$, $\tau^V = \det(m_{i,j}^V)_{i,j=0}^{m-1}$,

$$c_3 = q^{\frac{n(n-1)}{2}} \frac{(q^{-n} \frac{k}{a_3}, q)_n (a_3, q)_n (q^{-m-n+1} \frac{a_3}{a_1}, q)_n (q^{m+1} \frac{a_1 a_3}{k}, q)_n}{(\frac{k}{a_2 a_3}, q)_n (\frac{q a_3}{a_2}, q)_n (q^{-m-n+1} \frac{a_3}{a_1}, q^2)_n (q^{m+n} \frac{a_1 a_3}{k}, 1)_n},$$

$$c_4 = \frac{(q^{-n} \frac{k}{a_4}, q)_n (a_4, q)_n}{(\frac{k}{a_2 a_4}, 1)_n (\frac{q a_4}{a_2}, q^2)_n}, \quad c_i = \frac{(\frac{k}{q a_i}, 1)_n (a_i, 1)_n}{(\frac{k}{a_2 a_i}, q)_n (\frac{q a_i}{a_2}, q)_n}, \quad (i = 5, 6),$$

$(x, v)_n = \prod_{i=0}^{n-1} \vartheta_p(xv^i)$ and $(c'_3, c'_4, c'_5, c'_6) = (c_4, c_3, c_5, c_6) \Big|_{(m, n, a_1, \dots, a_6) \mapsto (n, m, \frac{k}{a_2}, \frac{k}{a_1}, \frac{k}{a_4}, \frac{k}{a_3}, \frac{k}{a_5}, \frac{k}{a_6})}$.

Proof. Since $\phi_i(a_4) = \delta_{i,0}$ ($i \geq 0$), we have

$$\frac{U(a_4)}{\text{const.}} = \det(m_{i,j+1}^U)_{i,j=0}^{n-1} = T_{a_2}^{-1} T_{a_4}(\tau^U). \quad (71)$$

Using the symmetry of $U(x)$ in parameters a_3, \dots, a_6 , the first relation of eq.(70) follows. The second relation is similar. \square

The determinant expressions for the special solutions have been known for various (discrete) Painlevé equations (see [7] [13] for example). Our method using Padé interpolation gives a simple and direct way to obtain them.

A Affine Weyl group actions

Here we give a derivation of the Painlevé equation (39), (40) from the affine Weyl group actions.[8] [20]

Define multiplicative transformations s_{ij} , c , μ_{ij} , ν_{ij} ($1 \leq i \neq j \leq 8$) acting on variables $h_1, h_2, u_1, \dots, u_8$ as

$$\begin{aligned} s_{ij} &= \{u_i \leftrightarrow u_j\}, & c &= \{h_1 \leftrightarrow h_2\}, \\ \mu_{ij} &= \{h_1 \mapsto \frac{h_1 h_2}{u_i u_j}, \quad u_i \mapsto \frac{h_2}{u_j}, \quad u_j \mapsto \frac{h_2}{u_i}\}, \\ \nu_{ij} &= \{h_2 \mapsto \frac{h_1 h_2}{u_i u_j}, \quad u_i \mapsto \frac{h_1}{u_j}, \quad u_j \mapsto \frac{h_1}{u_i}\}. \end{aligned} \quad (72)$$

These actions generate the affine Weyl group of type $E_8^{(1)}$ with the following simple reflections:

$$\begin{array}{ccccccc} & & s_{12} & & & & \\ & & | & & & & \\ c & - & \mu_{12} & - & s_{23} & - & s_{34} & - & \dots & - & s_{78} & . \end{array} \quad (73)$$

We extend the actions bi-rationally on variables (f, g) . The nontrivial actions are as follows:

$$c(f) = g, \quad c(g) = f, \quad \mu_{ij}(f) = \tilde{f}, \quad \nu_{ij}(g) = \tilde{g}, \quad (74)$$

where, $\tilde{f} = \tilde{f}_{ij}$ and $\tilde{g} = \tilde{g}_{ij}$ are rational functions in (f, g) defined by

$$\frac{\tilde{f} - \mu_{ij}(f_i)}{\tilde{f} - \mu_{ij}(f_j)} = \frac{(f - f_i)(g - g_j)}{(f - f_j)(g - g_i)}, \quad \frac{\tilde{g} - \nu_{ij}(g_i)}{\tilde{g} - \nu_{ij}(g_j)} = \frac{(g - g_i)(f - f_j)}{(g - g_j)(f - f_i)}, \quad (75)$$

$(f_i, g_i) = (f_\star(u_i), g_\star(u_i))$, and

$$f_\star(z) = \frac{\vartheta_p(\frac{d_2}{z}, \frac{h_1}{d_2 z})}{\vartheta_p(\frac{d_1}{z}, \frac{h_1}{d_1 z})}, \quad g_\star(z) = \frac{\vartheta_p(\frac{d_2}{z}, \frac{h_2}{d_2 z})}{\vartheta_p(\frac{d_1}{z}, \frac{h_2}{d_1 z})}, \quad (76)$$

as in eq.(10). As a rational function of (f, g) , \tilde{f} is characterized by the following properties:

(i) it is of degree $(1, 1)$ with indeterminate points (f_i, g_i) , (f_j, g_j) , (ii) it maps generic points on the elliptic curve $(f_\star(z), g_\star(z))$ to $\frac{\vartheta_p(\frac{d_2}{z}, \frac{h_1 h_2}{d_2 z u_1 u_2})}{\vartheta_p(\frac{d_1}{z}, \frac{h_1 h_2}{d_1 z u_1 u_2})}$. Using this geometric characterization,

we have

$$\mu_{ij} \left\{ \frac{\mathcal{F}_f(\frac{h_1 z}{h_2})}{\mathcal{F}_f(z)} \right\} = \frac{\vartheta_p(\frac{u_i}{z}, \frac{u_j}{z})}{\vartheta_p(\frac{h_2}{u_i z}, \frac{h_2}{u_j z})} \frac{\mathcal{F}_f(\frac{h_1 z}{h_2})}{\mathcal{F}_f(z)}, \quad \text{for } g = g_\star(z), \quad (77)$$

where the functions $\mathcal{F}_f(z)$ (and $\mathcal{G}_g(z)$) are defined in a similar way as eq.(11)

$$\mathcal{F}_f(z) = \vartheta_p\left(\frac{d_1}{z}, \frac{h_1}{d_1 z}\right)f - \vartheta_p\left(\frac{d_2}{z}, \frac{h_1}{d_2 z}\right), \quad \mathcal{G}_g(z) = \vartheta_p\left(\frac{d_1}{z}, \frac{h_2}{d_1 z}\right)g - \vartheta_p\left(\frac{d_2}{z}, \frac{h_2}{d_2 z}\right). \quad (78)$$

Let us consider the following compositions [8]

$$r = s_{12}\mu_{12}s_{34}\mu_{34}s_{56}\mu_{56}s_{78}\mu_{78}, \quad T = rcrc. \quad (79)$$

Their actions on variables (h_i, u_i) are given by

$$\begin{aligned} r(h_1) &= v h_2, & r(h_2) &= h_2, & r(u_i) &= \frac{h_2}{u_i}, \\ T(h_1) &= q h_1 v^2, & T(h_2) &= q^{-1} h_2 v^2, & T(u_i) &= u_i v, \end{aligned} \quad (80)$$

where $v = q h_2 / h_1$, $q = h_1^2 h_2^2 / (u_1 \cdots u_8)$. From eq.(77) and $r(\frac{h_1}{h_2}) = \frac{q h_2}{h_1}$, the evolution $T(f) = rcrc(f) = r(f)$ is determined as

$$\frac{\mathcal{F}_f(z)}{\mathcal{F}_f(\frac{h_1 z}{h_2})} \frac{T(\mathcal{F}_f)(\frac{q h_2 z}{h_1})}{T(\mathcal{F}_f)(z)} = \prod_{i=1}^8 \frac{\vartheta_p(\frac{u_i}{z})}{\vartheta_p(\frac{h_2}{u_i z})}, \quad \text{for } g = g_*(z). \quad (81)$$

Similarly, since $cTc = T^{-1}$, $T^{-1}(g)$ is determined by

$$\frac{\mathcal{G}_g(z)}{\mathcal{G}_g(\frac{h_2 z}{h_1})} \frac{T^{-1}(\mathcal{G}_g)(\frac{q h_1 z}{h_2})}{T^{-1}(\mathcal{G}_g)(z)} = \prod_{i=1}^8 \frac{\vartheta_p(\frac{u_i}{z})}{\vartheta_p(\frac{h_1}{u_i z})}, \quad \text{for } f = f_*(z). \quad (82)$$

By a re-scaling of variables $(h_i, u_i, d_i) = (\kappa_i \lambda^2, \xi_i \lambda, c_i \lambda)$ with $\lambda = (h_1^3 h_2^{-1})^{\frac{1}{4}}$, we have $\mathcal{F}_f(z) = F_f(\frac{z}{\lambda})$, $T(\mathcal{F}_f)(z) = T(F_f)(\frac{\kappa_1}{\kappa_2} \frac{z}{\lambda})$ and so on, since $T(\lambda) = \frac{h_2}{h_1} \lambda$. Then the above equations take the form (39), (40), by putting $z = \lambda x$.

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